General constructs in condensed matter physics

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(Dated: January 8, 2023)

PACS numbers: 75.60.Ch and 72.25.Ba.

I. CAUSAL GREENS' FUNCTION FOR THE WAVE EQUATION

We look at the inhomogeneous wave equation, which reads as follows ;

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = -4\pi f(\mathbf{x}, t), \tag{1}$$

and we have, by definition, the Greens function :

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{x}, t; \mathbf{x}^{'}, t^{'}) = -4\pi\delta^3(\mathbf{x} - \mathbf{x}^{'})\delta(t - t^{'}). \tag{2}$$

Transforming by a Fourier transform, we have the following :

$$g(\mathbf{k}, \omega) = \frac{1}{4\pi^3}\left(\frac{1}{\mathbf{k}^2 - \omega^2/c^2 - i\epsilon}\right) \tag{3}$$

This is the causal Greens function because we have chosen the incremental complex offset from the time axis. In order to transform back to the time domain, we have the following :

$$G(\mathbf{x}, t; \mathbf{x}^{'}, t^{'}) = \int d^3\mathbf{k} d\omega e^{i\omega(t-t^{'})} e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}^{'})} g(\mathbf{k}, \omega) \tag{4}$$

$$\tag{5}$$

We must now convert this 3 D integral in **k** space into a series of convolution of 1 D integral in order to use complex-valued contours to evaluate this expression in a closed form.

$$= \int_0^\infty dk k^2 \int_0^\pi \sin\theta \int_0^{2\pi} d\phi d\omega e^{i\omega(t-t^{'})} e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}^{'})} \frac{1}{4\pi^3}\left(\frac{1}{\mathbf{k}^2 - \omega^2/c^2 - i\epsilon}\right) \tag{6}$$

$$\int_0^\infty dk k^2 \int_1^{-1} -d\mu \int_0^{2\pi} d\phi d\omega e^{i\omega(t-t^{'})} e^{ik.\overline{x}\mu} \frac{1}{4\pi^3}\left(\frac{1}{\mathbf{k}^2 - \omega^2/c^2 - i\epsilon}\right) \tag{7}$$

where we define $\mu = \cos(\theta)$. In order to evaluate this integral, let us appeal to the Cauchy theorem as applied to integration, as follows :

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{N} Res_{z=a_k} f(z), \tag{8}$$

which is the statement of the Cauchy theorem. The poles of the analytic complex-valued function are given by a_k. This is the residue theorem and it transforms the x-represented function to the z-representation, and integrates around each term in the Laurent expansion, to the leave the one coefficient c_{-1}. Apply the theorem to for the first

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(z=ω/c) of the two poles (in the limit $\epsilon \to 0^+$) ; z$\to$ 0.

$$Res[f(z)] = \lim_{z\to\omega/c} \frac{(z-\omega/c)z^2 e^{iz\|\overline{x}\|\mu}}{z^2-\omega^2/c^2} \tag{9}$$

$$= \lim_{z\to\omega/c} \frac{z^2 e^{iz\|\overline{x}\|\mu}}{z+\omega/c} \tag{10}$$

$$= \frac{\omega}{2c} e^{i\frac{\omega}{c}\|\overline{x}\|\mu}, \tag{11}$$

where we define $\overline{x} = \mathbf{x} - \mathbf{x}^{'}$. The contribution of the other pole to residue to the Laurent expanded integral gives :

$$Res_{z\to-\omega/c}[f(z)] = \frac{-\omega}{2c} e^{-i\frac{\omega}{c}\|\overline{x}\|\mu}, \tag{12}$$

and, combining the results we have the following expression :

$$g(x,t;x^{'},t^{'}) = \int d\omega e^{i\omega(t-t^{'})} \int_0^{2\pi} \int_0^{\pi} -d\mu \frac{\omega}{2c} \left(e^{i\frac{\omega}{c}\|\overline{x}\|\mu} - e^{-i\frac{\omega}{c}\|\overline{x}\|\mu} \right) 2\pi i \tag{13}$$

which gives us :

$$g(x,t;x^{'},t^{'}) = \int d\omega e^{i\omega(t-t^{'})} \frac{(2\cos\frac{\omega}{c}\|\overline{x}\|)}{\|\overline{x}\|} \tag{14}$$

$$= \frac{1}{|\mathbf{x}-\mathbf{x}^{'}|} \delta(t - t^{'} + \frac{\|\mathbf{x}-\mathbf{x}^{'}\|}{c}) \tag{15}$$

$$\tag{16}$$